ADDENDUM TO "SUBSET CURRENTS ON FREE GROUPS"

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ABSTRACT. In a paper with Nagnibeda [9] we proved that for any $N \geq 2$ the set $\mathcal{S}\mathrm{Curr}_r(F_N)$ of all rational subset currents is dense in the space $\mathcal{S}\mathrm{Curr}(F_N)$ of subset currents on F_N. That proof was indirect and relied on deep results of Bowen and Elek about unimodular graph measures. In this note we give a direct proof that rational subset currents are dense in $\mathcal{S}\mathrm{Curr}(F_N)$, via an explicit combinatorial construction. As an application of this method, we answer one of the questions (Problem 10.11) posed in [9]. Thus we prove that if a nonzero $\mu \in \mathcal{S}\mathrm{Curr}(F_N)$ has all weights with respect to some marking being integers, then μ is the sum of finitely many "counting" currents corresponding to nontrivial finitely generated subgroups of F_N.

1. Introduction

In a paper with Tatiana Nagnibeda [9] we introduced and studied the notion of a *subset current* on a free group F_N. This concept is motivated by that of a *geodesic current*. Geodesic currents on F_N are measures that generalize conjugacy classes of nontrivial elements of F_N. The space $Curr(F_N)$ of all geodesic currents on F_N turns out to be highly useful in the study of the dynamics and geometry of $\mathrm{Out}(F_N)$ and of the Culler-Vogtmann Outer space, particularly via the use of the "geometric intersection form constructed in [8]. See [9] for an extended discussion and [1, 5] for examples of new such applications. Similarly, the notion of a subset current is a measure-theoretic analog of the conjugacy class of a nontrivial finitely generated subgroup of F_N. For a free group F_N let $\mathfrak{C}_N$ be the space of all closed subsets $S \subseteq \partial F_N$ such that S consists of at least two elements. The space $\mathfrak{C}_N$ comes equipped with a natural topology (see Section 2 below and [9] for details) such that $\mathfrak{C}_N$ is a locally compact totally disconnected Hausdorff topological space. The action of F_N on ∂F_N by translations extends to a natural translation action of F_N on $\mathfrak{C}_N$ by homeomorphisms. A *subset current* on F_N is a positive Borel measure μ on $\mathfrak{C}_N$ such that μ is finite on compact subsets and is F_N-invariant. The space $\mathcal{S}\mathrm{Curr}(F_N)$ of all subset currents on F_N comes equipped with a natural weak-* topology and a natural action of $\mathrm{Out}(F_N)$ by continuous $\mathbb{R}_{\geq 0}$-linear transformations.

Given a nontrivial finitely generated subgroup $H \leq F_N$, there is a naturally associated *counting* subset current $\eta_H \in \mathcal{S}\mathrm{Curr}(F_N)$. The limit set $\Lambda(H) \subseteq \partial F_N$ is a closed F_N-invariant subset of ∂F_N and, since $H \neq \{1\}$, we have $\Lambda(H) \in \mathfrak{C}_N$. Moreover, for any $g \in F_N$ $\Lambda(gHg^{-1}) = g\Lambda(H)$. If H is equal to its commensurator $Comm_{F_N}(H)$, we define $\eta_H := \sum_{H_1 \in [H]} \delta_{\Lambda(H_1)}$, where $[H]$ is the conjugacy class of H in F_N. For an arbitrary nontrivial finitely generated subgroup $H \leq F_N$ it is known that $m := [Comm_{F_N}(H) : H] < \infty$ and that $Comm_{F_N}(H)$ is equal to its own commensurator in F_N. Then we define $\eta_H := m\, \eta_{Comm_{F_N}(H)}$. It is shown in

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[9] that η_H is indeed a subset current on F_N. One can also equivalently describe η_H in more combinatorial terms, using Stallings core graphs, see [9] and Proposition-Definition 2.5 below. A subset current $\mu \in \mathcal{S}\mathrm{Curr}(F_N)$ is called *rational* if $\mu = c\eta_H$ for some $c \ge 0$ and some nontrivial a finitely generated $H \le F_N$. Denote by $\mathcal{S}\mathrm{Curr}_r(F_N)$ the set of all rational subset currents on F_N. One of the main results of [9] is:

Theorem 1.1. *Let $N \ge 2$ be an integer. Then $\mathcal{S}Curr_r(F_N)$ is a dense subset of $\mathcal{S}Curr(F_N)$.*

Theorem 1.1 generalizes a well-known similar result [7, 10] for $\mathrm{Curr}(F_N)$, but the case of $\mathcal{S}\mathrm{Curr}(F_N)$ is considerably more difficult. The proof of Theorem 1.1 in [9] is indirect and relies on deep work of Bowen and Elek about "unimodular graph measures", that is, measures on spaces of rooted graphs that are invariant, in the appropriate sense, with respect to root-change. Namely, given a free basis A of F_N and the Cayley graph X_A of F_N with respect to A, in [9] we relate subset currents to root-change invarinat measures on the space $\mathcal{T}_1(X_A)$ of all infinite subtrees Y of X_A without degree-one vertices such that Y contains the vertex 1 of X_A. For the latter space of measures one can use the results of Bowen [2, 3] and Elek [4] about weakly approximating these measures by sequences of finite graphs and eventually conclude that $\mathcal{S}\mathrm{Curr}_r(F_N)$ is dense in $\mathcal{S}\mathrm{Curr}(F_N)$. In the present note we give a direct combinatorial proof of Theorem 1.1, bypassing the "unimodular graph measures" results. The proof shares some similarities with the approach of Elek, but is more directly inspired by the ideas about approximating $\mathrm{Curr}(F_N)$ by a sequence of finite-dimensional rational polyhedra, that we used in our earlier work [6, 7] on geodesic currents on F_N. We construct a similar approximation of $\mathcal{S}\mathrm{Curr}(F_N)$ by finite dimensional polyhedra here. A key step in the proof of Theorem 1.1 is obtaining the "Integral weight realization theorem", see Theorem 4.3 below, which says that all integer points (weight systems) in these polyhedra can be realized as weight systems coming from finite Stallings core graphs. As an application of Theorem 4.3 we solve Problem 10.11 from [9] and prove in Theorem 4.6 below that any nonzero $\mu \in \mathcal{S}\mathrm{Curr}(F_N)$, such that all weights for μ with respect to some marking on F_N are integers, has the form $\mu = \eta_{H_1} + \cdots + \eta_{H_k}$ for some $k \ge 1$ and some nontrivial finitely generated subgroups $H_1, \dots, H_k \le F_N$.

2. Background

We will use the same notations, conventions and definitions as in [9] and only briefly recall some of them here. If Y is a graph, we denote by EY the set of oriented edges of Y. For $e \in EY$ $o(e)$ is the initial vertex of e, $t(e)$ is the terminal vertex of e and $e^{-1} \in EY$ is the inverse edge of e

2.1. **The space $\mathfrak{C}_N$.** Let F_N be a free group of finite rank $N \ge 2$. The space $\mathfrak{C}_N$ consists of all closes subsets $S \subseteq \partial F_N$ such that S consists of at least two points. We topologize $\mathfrak{C}_N$ by choosing a visual metric d on ∂F_N and then using the Hausdorff distance between closed subsets of ∂F_N to metrize $\mathfrak{C}_N$. This metric topology on $\mathfrak{C}_N$ does not depend on the choice of a visual metric on ∂F_N and turns $\mathfrak{C}_N$ into a locally compact totally disconnected Hausdorff topological space. The topology on $\mathfrak{C}_N$ can be described more explicitly in terms of the "subset cylinders". Given a marking $\alpha : F_N \xrightarrow{\sim} \pi_1(\Gamma)$ (where Γ is a finite connected graph without degree-one and degree-two vertices), let $X = \widetilde{\Gamma}$, taken with the simplicial metric, where every

edge has length 1. Then α induces a quasi-isometry between F_N and X and hence gives an identification, via an F_N-equivariant homeomorphism, between ∂F_N and ∂X. As in [9], we denote by $\mathcal{K}_\Gamma$ the set of all finite non-degenerate subtrees $K \subseteq X$. If e is an oriented edge of X, we denote by $Cyl_X(e)$ the set of all $\xi \in \partial F_N$ such that the geodesic from $o(e)$ to ξ in X starts with e. Thus $Cyl_X(e) \subseteq \partial F_N$ is a compact-open subset of ∂F_N. Now let $K \in \mathcal{K}_\Gamma$. Let $e_1, \dots, e_n \in EX$ be all the terminal edges of K. We define the *subset cylinder* $\mathcal{S}Cyl_\alpha(K) \subseteq \mathfrak{C}_N$ as the set of all $S \in \mathfrak{C}_N$ such that $S \subseteq \cup_{i=1}^n Cyl_X(e_i)$ and such that for each $i = 1, \dots, n$ $S \cap Cyl_X(e_i) \neq \emptyset$. Then $\mathcal{S}Cyl_\alpha(K)$ is a compact-open subset of $\mathfrak{C}_N$ and the family $\{\mathcal{S}Cyl_\alpha(K) | K \in \mathcal{K}_\Gamma\}$ forms a basis for the topology on $\mathfrak{C}_N$ defined above.

Denote by $q(e)$ the set of all oriented edges e' in X such that e, e' is a reduced edge-path in X. For any set B we denote by $P_+(B)$ the set of all nonempty subsets of B. The following basic fact plays a key role in the theory of subset currents:

Lemma 2.1. *[c.f. Lemma 3.5 in* [9]*] Let $K \in \mathcal{K}_\Gamma$ and let $e_1, \dots, e_n$ be all the terminal edges of K. Then for every $i = 1, \dots, n$ we have*

$$\mathcal{S}Cyl_\alpha(K) = \sqcup_{U \in P_+(q(e_i))} \mathcal{S}Cyl_\alpha(K \cup U).$$

2.2. **Subset currents.** A *subset current* on F_N is a positive Borel measure measure μ on $\mathfrak{C}_N$ which is F_N-invariant and locally finite, that is, finite on all compact subsets of $\mathfrak{C}_N$.

The set of all subset currents on F_N is denoted $\mathcal{S}\mathrm{Curr}(F_N)$. The space $\mathcal{S}\mathrm{Curr}(F_N)$ is endowed with the natural weak-* topology of point-wise convergence of integrals of continuous functions. The weak-* topology on $\mathcal{S}\mathrm{Curr}(F_N)$ can be described in more concrete terms:

Let $\mu, \mu_n \in \mathcal{S}\mathrm{Curr}(F_N)$. Then $\lim_{n\to\infty} \mu_n = \mu$ in $\mathcal{S}\mathrm{Curr}(F_N)$ if and only if for every finite non-degenerate subtree K of X we have

$$\lim_{n\to\infty} \mu_n(\mathcal{S}Cyl_\alpha(K)) = \mu(\mathcal{S}Cyl_\alpha(K)).$$

For $K \in \mathcal{K}_\Gamma$ and $\mu \in \mathcal{S}\mathrm{Curr}(F_N)$ denote $\langle K, \mu \rangle_\alpha := mu(\mathcal{S}Cyl_\alpha(K))$ and call this quantity the *weight* of K in μ. If $\mu, \mu' \in \mathcal{S}\mathrm{Curr}(F_N)$ satisfy $\langle K, \mu \rangle_\alpha = \langle K, \mu' \rangle_\alpha$ for all $K \in \mathcal{K}_\Gamma$, then $\mu = \mu'$.

Note that if $K \in \mathcal{K}_\Gamma$ and $g \in F_N$ then $g\mathcal{S}Cyl_\alpha(K) = \mathcal{S}Cyl_\alpha(gK)$. Hence for any $\mu \in \mathcal{S}\mathrm{Curr}(F_N)$, $g \in F_N$ and $K \in \mathcal{K}_\Gamma$ we have $\mu(\mathcal{S}Cyl_\alpha(K)) = \mu(g\mathcal{S}Cyl_\alpha(K))$, so that $\langle K, \mu \rangle_\alpha = \langle gK, \mu \rangle_\alpha$. For a given finite subtree K of X, we denote the F_N-translation class of K by $[K]$ (so that $[K]$ consists of all the translates of K by elements of F_N). We put $\langle [K], \mu \rangle_\alpha := \langle K, \mu \rangle_\alpha$ and call it the *weight* of $[K]$ in μ.

Lemma 2.1 immediately implies (c.f. Proposition 3.11 in [9]):

Proposition 2.2 (Kirchhoff formulas for weights)**.** *Let K be a finite non-degenerate subtree of X. Let e be one of the terminal edges of K and let $\mu \in \mathcal{S}Curr(F_N)$. Then*

$$\langle K, \mu \rangle_\alpha = \sum_{U \in P_+(q(e))} \langle K \cup U, \mu \rangle_\alpha. \tag{$\bigstar$}$$

2.3. **Γ-graphs.** Let $\alpha : F_N \xrightarrow{\sim} \pi_1(\Gamma)$ be a marking. A Γ-*graph* is a graph Δ together with a graph morphism $\tau : \Delta \to \Gamma$. For a vertex $x \in V\Delta$ we sat that the *type* of x is the vertex $\tau(x) \in V\Gamma$. Similarly, for an oriented edge $e \in E\Gamma$ the *type* of e, or the *label* of e is the edge $\tau(e)$ of Γ. Every covering of Γ has a canonical Γ-graph structure. In particular, Γ itself is a Γ-graph and so is the universal cover $\widetilde{\Gamma}$ of Γ. Also, every subgraph of a Γ-graph is again a Γ-graph.

Let $\tau_1 : \Delta_1 \to \Gamma$ and $\tau_2 : \Delta_2 \to \Gamma$ be Γ-graphs. A graph-map $f : \Delta_1 \to \Delta_2$ is called a Γ-*map*, or Γ-*morphism*, if it respects the labels of vertices and edges, that is if $\tau_1 = \tau_2 \circ f$.

A Γ-graph Δ is *folded* if the labeling map $\tau : \Delta \to \Gamma$ is an immersion, that is, if τ is locally injective.

Definition 2.3 (Link of a vertex)**.** Let Δ be a Γ-graph. For a vertex $x \in V\Delta$ denote by $Lk_\Delta(x)$ (or just by $Lk(x)$) the function

$$Lk_\Delta(x) : E\Gamma \to \mathbb{Z}_{\geq 0}$$

where for every $e \in E\Gamma$ the value $(Lk_\Delta(x))\,(e)$ is the number of edges of Δ with origin x and label e.

Thus a Γ-graph Δ is folded if and only if for every vertex $x \in V\Delta$ and every $e \in E\Gamma$ we have

$$(Lk_\Delta(x))\,(e) \leq 1.$$

If Δ is folded, we will also think of $Lk_\Delta(x)$ as a subset of $E\Gamma$ consisting of all those $e \in E\Gamma$ with $(Lk_\Delta(x))\,(e) = 1$, that is, of all $e \in E\Gamma$ such that there is an edge in Δ with origin x and label e.

We say that a nonempty finite Γ-graph Δ is *cyclically reduced* if Δ is folded and every vertex of Δ has degree ≥ 2.

If $\tau : \Delta \to \Gamma$ is a cyclically reduced Γ-graph, then $W := \tau_\#(\pi_1(\Delta)) \leq \pi_1(\Gamma)$ is a finitely generated subgroup of $\pi_1(\Gamma)$. Recall that we also have a marking $\alpha : F_N \xrightarrow{\simeq} \pi_1(\Gamma)$. We say that the subgroup $H := \alpha^{-1}(W) \leq F_N$ is *represented* by Δ. The conjugacy class of $[H]$ in F_N does not change if we replace α by an equivalent marking.

Definition 2.4 (Occurrence)**.** Let $K \subseteq \widetilde{\Gamma}$ be a finite non-degenerate subtree (recall that $\widetilde{\Gamma}$ and all of its subgraphs have canonical Γ-graph structure).

Let Δ be a finite cyclically reduced Γ-graph. An *occurrence* of K in Δ is a Γ-morphism $\mathfrak{O} : K \to \Delta$ such that for every vertex x of K of degree at least 2 in K we have $Lk_K(x) = Lk_\Delta(\mathfrak{O}(x))$.

We denote the number of all occurrences of K in Δ by $\langle K; \Delta\rangle_\Gamma$, or just $\langle K; \Delta\rangle$.

In topological terms, a Γ-morphism $\mathfrak{O} : K \to \Delta$ is an occurrence of K in Δ if $\mathfrak{O}$ is an immersion and if $\mathfrak{O}$ is a covering map at every point $x \in K$ (including interior points of edges) except for the degree-1 vertices of K. That is, for every $x \in K$, other than a degree-1 vertex of K, $\mathfrak{O}$ maps a small neighborhood of x in K homeomorphically *onto* a small neighborhood of $\mathfrak{O}(x)$ in Δ.

We need the following key fact from [9]:

Proposition-Definition 2.5. Let $\alpha : F_N \to \pi_1(\Gamma)$ be a marking on F_N and let $X = \widetilde{\Gamma}$. Recall that $\mathcal{K}_\Gamma$ is the set of all non-degenerate finite simplicial subtrees of X. Let $\tau : \Delta \to \Gamma$ be a finite cyclically reduced Γ-graph.

Then there is a unique generalized current $\mu_\Delta \in \mathcal{S}\mathrm{Curr}(F_N)$ such that for every $K \in \mathcal{K}_\Delta$

$$\langle K, \mu_\Delta\rangle_\alpha = \langle K; \Delta\rangle_\Gamma$$

Moreover, if Δ is also connected, then $\mu_\Delta = \eta_H$, where $H \leq F_N$ is the finitely generated subgroup of F_N represented by Δ.

3. More on cylinders and Kirchhoff-type formulas

Convention 3.1. From now and for the remainder of this paper, unless specified otherwise, we fix a marking $\alpha : F_N \to \pi_1(\Gamma)$. Put $X = \widetilde{\Gamma}$. We also equip X with the simplicial metric d, by giving each edge of X length 1.

Let $K \subseteq X$ be a nondegenerate finite subtree and let e be a terminal edge of K. For an integer $m \geq 1$ we say that a finite nondegenerate subtree $U \subseteq X$ is (K, e, m)-*admissible* if:

(1) We have $K \cap U = \{t(e)\}$.
(2) For every terminal vertex v of U such that $v \neq t(e)$ we have $d(t(e), v) = m$.

For $m = 0$ we also say that the degenerate tree $U = \{t(e)\}$ is $(K, e, 0)$-*admissible*.

For $m \geq 1$ we denote by $\mathcal{B}(K, e, m)$ the set of all U such that U is (K, e, m)-admissible. Thus $P_+(q(e)) = \mathcal{B}(K, e, 1)$.

Lemma 2.1 easily implies:

Corollary 3.2. *Let $K \subseteq X$ be a nondegenerate finite subtree and let e be a terminal edge of K. Then for every integer $m \geq 1$ we have*

$$\mathcal{S}Cyl_\alpha(K) = \sqcup_{U \in \mathcal{B}(K,e,m)} \mathcal{S}Cyl_\alpha(K \cup U).$$

Definition 3.3 (Round graph)**.** For an integer $r \geq 1$, we say that a finite subtree K of X is a *round graph* of *grade* r in X if there exists a (necessarily unique) vertex v of K such that for every terminal vertex u of K we have $d(v, u) = r$.

Let $K \subseteq X$ be a nondegenerate finite subtree and let v be a vertex of K (possibly a terminal vertex). We denote by $R(K, v)$ the maximum of $d(v, v')$ where v' varies over all terminal vertices of K. The fact that K is nondegenerate means that $R(K, v) \geq 1$.

Let $e_1, \dots, e_n$ be the terminal edges of K and let $r \geq R(K, v)$ be an integer. We say that an n-tuple $\mathcal{T} = (U_1, \dots, U_n)$ of finite subtrees U_i of X is (K, v, r)-*admissible* if for each $i = 1, \dots, n$ the tree U_i is (K, e_i, m_i)-admissible, where $m_i = r - d(v, t(e_i))$. Note that if $\mathcal{T} = (U_1, \dots, U_n)$ is (K, v, r)-admissible and $K' = K \cup U_1 \cdots \cup U_n$ then for every terminal vertex u of K' we have $d(v, u) = r$. Thus K' is a round graph of grade r with center v.

Corollary 3.2 directly implies:

Corollary 3.4. *Let $K \subseteq X$ be a nondegenerate subtree with terminal edges $e_1, \dots, e_n$. Let v be a vertex of K and let $r \geq R(K, v)$ be an integer. Denote by $\mathcal{B}(K, v, r)$ the set of all (K, v, r)-admissible n-tuples. Then*

$$\mathcal{S}Cyl_\alpha(K) = \sqcup\{\mathcal{S}Cyl_\alpha(K \cup U_1 \cup \dots \cup U_n) | (U_1, \dots, U_n) \in \mathcal{B}(K, v, r)\}.$$

For a finite nondegenerate subtree $K \subseteq X$ we put $r(K)$ to be the mimimum of $R(K, v)$ where v varies over all vertices of K. We refer to $r(K)$ as the *radius* of K.

In view of finite additivity of subset currents, Corollary 3.2 and Corollary 3.4 immediately imply:

Corollary 3.5. *Let $K \subseteq X$ be a finite non-degenerate subtree of X and let $\mu \in \mathcal{S}Curr(F_N)$. Then:*

(1) *For any terminal edge e of K and any integer $m \geq 1$ we have*

$$\langle K, \mu \rangle_\alpha = \sum_{U \in \mathcal{B}(K,e,m)} \langle K \cup U, \mu \rangle_\alpha.$$

(2) *Let v be a vertex of K, let $e_1, \dots, e_n$ be the terminal edges of K and let $r \ge R(K,v)$ be an integer. Then have*
$$\langle K, \mu\rangle_\alpha = \sum_{(U_1,\dots,U_n)\in\mathcal{B}(K,v,r)} \langle K \cup U_1 \cdots \cup U_n, \mu\rangle_\alpha.$$

Recall that, as noted earlier, for part (2) above, if $(U_1,\dots,U_n) \in \mathcal{B}(K,v,r)$ and $K' = K \cup U_1 \cdots \cup U_n$ then for any terminal vertex u of K' we have $d(v,u) = r$, so that K' is a round graph of grade r in X. Thus part (2) of Corollary 3.5 implies that, for $\mu \in \mathcal{S}\mathrm{Curr}(F_N)$ and an integer $r \ge 1$, knowing the μ-weights of all round graphs of grade r uniquely determines the μ-weights of all the subtrees of radius $\le r$.

Definition 3.6 (Semi-round graph)**.** Let p be the mid-point of an edge e of X and let $r \ge 2$ be an integer. We say that a finite subtree J of X is a *semi-round graph* of *grade* r with *center* p if $e \in H$ and if for every terminal vertex u of J we have $d(p,u) = r - \frac{1}{2}$. Thus for every terminal vertex u of J belonging to the connected component of $J - \{p\}$ containing $o(e)$ we have $d(o(e), v) = r-1$. Similarly, for every terminal vertex u of J belonging to the connected component of $J - \{p\}$ containing $t(e)$ we have $d(t(e), v) = r - 1$.

Definition 3.7 (Child of a round graph)**.** Let $r \ge 2$ and let $K \subseteq X$ be a round graph of grade r in X centered at a vertex v of X. Let e be an edge of K with $o(e) = v$. Let p be the mid-point of e. We define a semi-round graph of grade r, centered at p, called the *e-child of K* and denoted K_e, as follows:

The graph K_e consists of all points $q \in K$ with $d(p,q) \le r - \frac{1}{2}$.

In other words, K_e is obtained from K by removing all those terminal vertices u of K and the terminal edges of K adjacent to these vertices such that the geodesic $[v,u]$ does not pass through the edge e.

Definition 3.7 is illustrated in Figure 1.

If H is a semi-round graph of grade r with center at the midpoint p of an edge e, then H can be enlarged to round graphs of grade r in two different "directions", namely to round graphs centered at $o(e)$ and at $t(e)$. This yields the following:

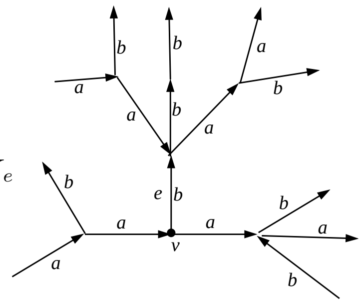
Proposition 3.8. *Let $J \subseteq X$ be a semi-round graph of grade $r \ge 2$ centered at the midpoint p of an edge e. Let J_0 and J_1 be the connected components of $J - \{p\}$ containing $o(e)$ and $t(e)$ accordingly. Let $e_1, \dots, e_n$ be all the terminal edges of J contained in J_0 and let $f_1, \dots, f_k$ be all the terminal edges of J contained in J_1. Let $\mathcal{B}_0$ be the set of all n-tuples of the form $(U_1, \dots, U_n)$ where each $U_i \in P_+(q(e_i))$, and let $\mathcal{B}_1$ be the set of all k-tuples of the form $(V_1, \dots, V_k)$ where each $V_j \in P_+(q(f_j))$.*

Then for any $\mu \in \mathcal{S}Curr(F_N)$ we have
$$\langle J, \mu\rangle_\alpha = \sum_{(U_1,\dots,U_n)\in\mathcal{B}_0} \langle J \cup U_1 \cdots \cup U_n, \mu\rangle_\alpha =$$
$$= \sum_{(V_1,\dots,V_k)\in\mathcal{B}_1} \langle J \cup V_1 \cdots \cup V_k, \mu\rangle_\alpha.$$

(Note that in the above summation each $J \cup U_1 \cdots \cup U_n$ is a round graph of grade r centered at $o(e)$ and each $J \cup V_1 \cdots \cup V_k$ is a round graph of grade r centered at $t(e)$.)

Proof. This statement is a direct corollary of Proposition 2.2. ∎

Child K_e



Round graph K

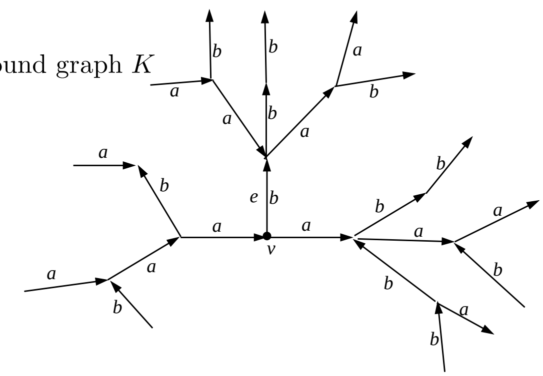


FIGURE 1. Child K_e of a round graph K of grade 3 with center v. Here $N = 2$, $F_2 = F(a, b)$ and Γ is the standard rose corresponding to the free basis $\{a, b\}$ of $F(a, b)$.

4. Finite-dimensional polyhedral approximations of $\mathcal{S}\mathrm{Curr}(F_N)$ and the integral weight realization theorem

Let $r \geq 2$ be an integer. Denote by $\mathcal{B}_{\Gamma,r}$ the set of all finite subtrees $K \subseteq X$ such that K is a round graph of grade r in X. Also, denote by $\mathbf{B}_{\Gamma,r}$ the set of all F_N-translation classes $[K]$ of trees $K \in \mathcal{B}_{\Gamma,r}$.

Denote by $\mathbf{J}_{\Gamma,r}$ the set of all F_N-translation classes $[J]$ of semi-round graphs $J \subseteq X$ of grade r.

Definition 4.1 (Approximating polyhedra)**.** Denote by $\mathcal{Q}_{\Gamma,r}$ the set of all functions $\vartheta : \mathcal{B}_{\Gamma,r} \to \mathbb{R}_{\geq 0}$ satisfyng the following properties:

(1) For every $K \in \mathcal{B}_{\Gamma,r}$ and every $g \in F_N$ we have $\vartheta(K) = \vartheta(gK)$.
(2) For every semi-round graph $J \subseteq X$ of grade r, in the notations of Proposition 3.8 we have
$$\sum_{(U_1,\dots,U_n)\in\mathcal{B}_0} \vartheta(J \cup U_1 \cdots \cup U_n) = \sum_{(V_1,\dots,V_k)\in\mathcal{B}_1} \vartheta(J \cup V_1 \cdots \cup V_k).$$

Note that since X is locally finite, there are only finitely many F_N-translation classes $[K]$ of trees $K \in \mathcal{B}_{\Gamma,r}$. Thus a point $\theta \in \mathcal{Q}_{\Gamma,r}$ can be viewed as a function from a finite set $\mathbf{B}_{\Gamma,r}$ to $\mathbb{R}_{\geq 0}$. Namely, if m is the cardinality of $\mathbf{B}_{\Gamma,r}$, we can view $\mathcal{Q}_{\Gamma,r}$ as a subset of $\mathbb{R}^m_{\geq 0}$, given by finitely many linear equations with integer coefficients coming from condition (2) in Definition 4.1.

The following lemma is a straightforward inductive corollary of Definition 2.4:

Lemma 4.2. *Let Δ be a finite cyclically reduced Γ-graph. Then for every integer $r \geq 1$*

$$\#V(\Delta) = \sum_{[K]\in \mathbf{B}_{\Gamma,r}} \langle K, \Delta\rangle_\alpha.$$

Recall, that, by definition, any Γ-graph Υ comes equipped with a "labelling" graph-map $\tau : \Upsilon \to \Gamma$.

The following statement is our key technical tool.

Theorem 4.3. *[Integral weight realization theorem] Let $r \geq 2$ and let $\vartheta \in \mathcal{Q}_{\Gamma,r}$ be such that for some $K_0 \in \mathcal{B}_{\Gamma,r}$ we have $\vartheta(K_0) > 0$. Suppose also that for every $K \in \mathcal{B}_{\Gamma,r}$ we have $\vartheta(K) \in \mathbb{Z}$. Then there exists a cyclically reduced (and possibly disconnected) finite Γ-graph Δ such that for every $K \in \mathcal{B}_{\Gamma,r}$ we have $\vartheta(K) = \langle K, \Delta\rangle_\alpha$.*

Proof. For each $[K] \in \mathbf{B}_{\Gamma,r}$ we choose a representative $K \in [K]$, so that $K \in \mathcal{B}_{\Gamma,r}$ and let $v = v_{[K]}$ be the center vertex of K. Thus K is a round graph of rank r centered at v. Denote $n_{[K]} := \vartheta(K)$. By assumption every $n_{[K]} \geq 0$ is an integer and there exists $K_0 \in \mathcal{B}_{\Gamma,r}$ such that $n_{[K_0]} \geq 0$.

For every $[K] \in \mathbf{B}_{\Gamma,r}$ we make $n_{[K]}$ copies $v_{[K],i}$ (where $i = 1, \dots, n_{[K]}$) of the vertex $v_{[K]}$ together with "half-links" of $v_{[K]}$ in K. That is for each $v_{[K],i}$ and for each edge e of K with $o(e) = v_{[K]}$ we attach a closed half-edge $[v_{[K],i}, p_{e,i}]$ at $v_{[K],i}$ representing a copy of the initial half of the edge e.

We refer to the points $p_{e,i}$ as *sub-vertices* and to the segments $[v_{[K],i}, p_{e,i}]$ as *sub-edges*. We endow each sub-vertex $p_{e,i}$ with a *decoration*, which is an ordered pair $(\tau(e), [K_e])$, where K_e is the e-child of K at v.

Let Ω_ϑ be the collection of all the decorated "half-links" obtained in this way. Thus Ω_ϑ consists of $M := \sum_{[K]\in \mathbf{B}_{\Gamma,r}} \vartheta(K) = \sum_{[K]\in \mathbf{B}_{\Gamma,r}} n_{[K]}$ "half-links".

Condition (2) in Definition 4.1 implies that for every semi-round graph $J \subseteq X$ of grade r with center p being a mid-point of an oriented edge e_J of X, the number of sub-vertices with decoration $(\tau(e_J), [J])$ is equal to the number of sub-vertices with decoration $(\tau(e_J^{-1}), [J])$.

For each $[J] \in \mathbf{J}_{\Gamma,r}$ as above we choose a matching (i.e. a bijection) between the set of subvertices in Ω_ϑ with decoration $(\tau(e_J), [J])$ and the set of sub-vertices with decoration $(\tau(e_J^{-1}), [J])$

We then identify each sub-vertex with decoration $(\tau(e_J), [J])$ with the corresponding to it under this matching subvertex with decoration $(\tau(e_J^{-1}), [J])$. We perform these identifications simultaneously for all $[J] \in \mathbf{J}_{\Gamma,r}$.

This gluing procedure is illustrated in Figure 2.

The resulting object Δ has a natural structure of a graph, where every vertex is of the form $v_{[K],i}$ and every edge is obtained by gluing two sub-edges along a sub-vertex; thus subvertices become mid-points of edges in Δ.

Moreover, Δ inherits a natural Γ-graph structure as well.

Indeed, an oriented edge f in Δ arises as the result of gluing a sub-edge $[v_{[K],i}, p_{e,i}]$ and a sub-edge $[v_{[K'],j}, p_{e',j}]$ by indentifying the sub-vertices $p_{e,i}$ and $p_{e',j}$ where $p_{e,i}$ is decorated by $(\tau(e), [K_e])$ and $p_{e',j}$ is decorated by $(\tau(e'), [K_{e'}])$ such that $[K_e] = [K_{e'}]$ and such that $\tau(e') = \tau(e)^{-1}$. In Δ we have $o(f) = v_{[K],i}$ and $t(f) = v_{[K'],j}$. We put $\tau(f) := \tau(e) \in E\Gamma$ and $\tau(f^{-1}) := \tau(e') = \tau(e)^{-1}$. Also, for each vertex $v_{[K],i}$ of Δ put $\tau(v_{[K],i} := \tau(v_{[K])}$.

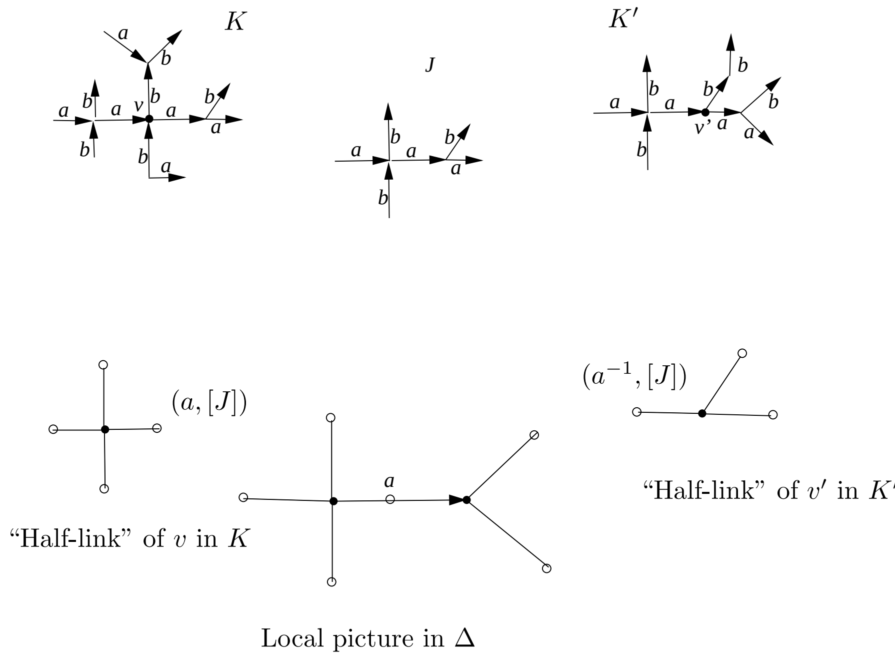


Figure 2. Illustration of the "gluing" procedure for constructing Δ in the proof of Theorem 4.3. Here $N = 2$, $F_2 = F(a, b)$ and Γ is the standard rose corresponding to the free basis $\{a, b\}$ of $F(a, b)$. Filled-in circles represent vertices and non-filled circles represent "sub-vertices".

This turns Δ into a nonempty Γ-graph. Moreover, by construction Δ is finite, folded and cyclically reduced and the number of vertices in Δ is equal to $M = \sum_{[K] \in \mathbf{B}_{\Gamma,r}} \vartheta(K) = \sum_{[K] \in \mathbf{B}_{\Gamma,r}} n_{[K]}$.

Note that by construction, for every vertex $v_{[K],i}$ of Δ we have $Lk_\Delta(v_{[K],i}) = Lk_K(v_{[K]})$. (Recall that $v_{[K]}$ is the center vertex of the round graph K). Moreover, by definition of a child of a round graph and using the fact that $r \geq 2$ we see that if $f = [v_{[K],i}, v_{[K'],j}]$ is an edge of Δ as in the preceding paragraph, then $Lk_\Delta(v_{[K'],j}) = Lk_K(t(e))$. Iteratively applying this crucial fact to the spheres of increasing radius around the center vertex in K, we see that for each vertex $v_{[K],i}$ of Δ as above, sending $v_{[K]}$ to $v_{[K],i}$ extends to a (necessarily unique) morphism of Γ-graphs $\mathfrak{O} : K \to \Delta$ with $\mathfrak{O}(v_{[K]} = v_{[K],i}$ such that $\mathfrak{O}$ is an occurrence of K in Δ in the sense of Definition 2.4.

Moreover, given a vertex u of Δ, there exists exactly one occurrence of a round graph of grade r in Δ that sends the center of that round graph to u (this occurrence corresponds to taking the ball of radius r in $\widetilde{\Delta}$ centered at a lift of u).

Thus, by construction, we see that for every $[K] \in \mathbf{B}_{\Gamma,r}$ the number of occurrences of $[K]$ in Δ is equal to $n_{[K]}$. This means that for every $K \in \mathcal{B}_{\Gamma,r}$ we have $\vartheta(K) = \langle K, \Delta \rangle_\alpha$, as required.

$\square$

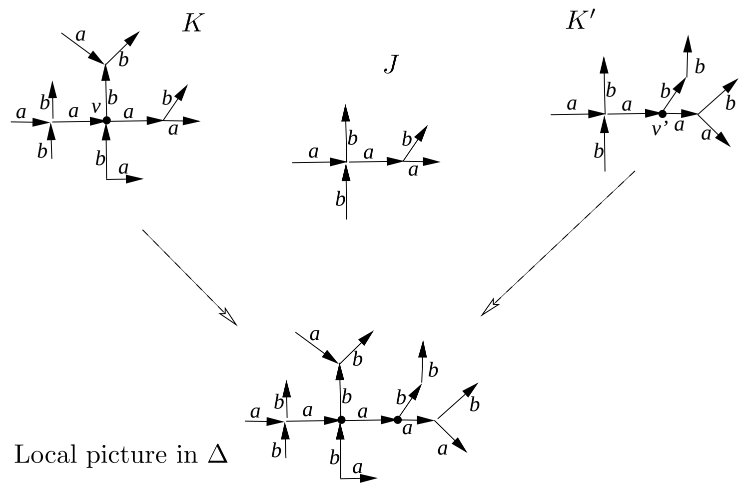


FIGURE 3. Illustration of the alternative "gluing" procedure for constructing Δ as in Remark 4.4.. Here $N = 2$, $F_2 = F(a,b)$ and Γ is the standard rose corresponding to the free basis $\{a,b\}$ of $F(a,b)$.

Remark 4.4. There is an alternative equivalent description of the graph Δ constructed in the proof of Theorem 4.3.

Namely, for every $[K] \in \mathbf{B}_{\Gamma,r}$ we make $n_{[K]} = \vartheta(K)$ copies $[K]_i$ (where $i = 1,\dots,n_{[K]}$) of K and denote the center vertex of $[K]_i$ by $v_{[K],i}$.

We then look at the set Ξ of all pairs $([K]_i, e)$ where $[K]_i$ is as above and e is an edge of K with $o(e) = v_{[K]}$, the center vertex of K. We endow each $([K]_i, e)$ with a "decoration" $(\tau(e), [K_e])$. Thus $[K_e]$ is a semi-round graph of grade r, which comes from the ball of radius $r - \frac{1}{2}$ in K centered at the midpoint of e.

Condition (2) in Definition 4.1 implies that for every semi-round graph $J \subseteq X$ of grade r with center p being a mid-point of an oriented edge e_J of X, the number elements of Ξ with decoration $(\tau(e_J), [J])$ is is equal to the number of elements of Ξ with decoration $(\tau(e_J^{-1}), [J])$.

For each $[J] \in \mathbf{J}_{\Gamma,r}$ as above we choose a matching between the set of elements of Ξ with decoration $(\tau(e_J), [J])$ and the set of elements of Ξ with decoration $(\tau(e_J^{-1}), [J])$.

Then we perform partial gluings on the disjoint union Ω of all $[K]_i$ (where $[K]$ varies over $\mathbf{B}_{\Gamma,r}$) as follows. Whenever $([K]_i, e)$ is matched with $([K']_j, e')$, it follows that the e-child K_e of K is (canonically) isomorphic as a Γ-graph to the e'-child $K'_{e'}$ of K'. (Recall that K_e is the ball of radius $r - \frac{1}{2}$ in K centered at the midpoint of e and that $K'_{e'}$ is the ball of radius $r - \frac{1}{2}$ in K' centered at the midpoint of e'). We glue the copy of K_e in $[K]_i$ to the copy $K'_{e'}$ in $[K']_j$ along the Γ-graph isomorphism between K_e and $K'_{e'}$. We perform these gluings simultaneously, on the disjoint union Ω of all $[K]_i$, as $[K]$ varies over $\mathbf{B}_{\Gamma,r}$. The result is a cyclically reduced finite Γ-graph which is the same as the Γ-graph Δ constructed in the proof of Theorem 4.3.

This alternative gluing procedure is illustrated in Figure 3.

Remark 4.5. Suppose that in Theorem 4.3 $\vartheta \in \mathcal{Q}_{\Gamma,r}$ has the property that whenever $\vartheta(K) > 0$ then $K \subseteq X$ is a geodesic segment of length $2r$ in X. Then for each such K the center vertex $v_{[K]}$ (which is the mid-point of this segment) has degree 2 in K and the proof of Theorem 4.3 produces a finite cyclically reduced graph Δ where every vertex has degree 2, so that Δ is a disjoint union of finitely many simplicial circles. One can use this fact to adapt the proof of Theorem 5.1 below to the case of ordinary geodesic currents and to produce a new proof (different from those given in [10, 7]) that the set of rational currents is dense in $\mathrm{Curr}(F_N)$.

The following statement provides a positive answer to Problem 10.11 in [9]:

Theorem 4.6. *Let $\mu \in \mathcal{S}Curr(F_N)$ be a nonzero subset current such that for every nondegenerate finite subtree $K \subseteq X$ we have $\langle K, \mu\rangle \in \mathbb{Z}$. Then there exists a finite cyclically reduced (possibly disconnected) Γ-graph Δ such that $\mu = \mu_\Delta$.*

Proof. Put $M := \sum_{[K]\in \mathbf{B}_{\Gamma,1}} \langle K, \mu\rangle_\alpha$.

For every $r \geq 2$ define the function $\theta_r : \mathcal{B}_{\Gamma,r} \to \mathbb{R}_{\geq 0}$ by $\theta_r(K) := \langle K, \mu\rangle_\alpha$, where $K \in \mathcal{B}_{\Gamma,r}$. Since μ is a nonzero subset current, we have that $\theta_r \in \mathcal{Q}_{\Gamma,r}$ for all $r \geq 2$.

Hence, by Theorem 4.3, for every $r \geq 1$ there exists a finite cyclically reduced Γ-graph Δ_r such that $\langle K, \Delta_r\rangle_\alpha = \langle K, \mu\rangle_\alpha$ for every $K \in \mathcal{B}_{\Gamma,r}$. Corollary 3.5 then implies that for every $r \geq 2$ and every finite nondegenerate subtree K of X of radius $\leq r$ we have $\langle K, \Delta_r\rangle_\alpha = \langle K, \mu\rangle_\alpha$. Hence, by Lemma 4.2, each graph Γ_r has exactly M vertices. There are only finitely many isomorphism types of finite cyclically reduced Γ-graphs with M vertices. Therefore there exist a finite cyclically reduced Γ-graph Δ such that for some sequence $r_n \to \infty$ as $n \to \infty$ the graph Δ_r is isomorphic, as Γ-graph, to Δ. For any finite nondegenerate subtree K of X there exists some r_n such that r_n is $\geq$ the radius of K. Therefore, by construction, for every finite nondegenerate subtree K of X we have $\langle K, \Delta\rangle_\alpha = \langle K, \mu\rangle_\alpha$. This implies that $\mu_\Delta = \mu$, as required.

□

5. Rational subset currents are dense

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Theorem 5.1. *Let $N \geq 2$. Then the set $\mathcal{S}Curr_r(F_N)$ of all rational subset currents is dense in $\mathcal{S}Curr(F_N)$.*

Proof. Let $\mu \in \mathcal{S}\mathrm{Curr}(F_N)$ be a nonzero subset current.

To show that μ can be approximated by rational subset currents it suffices to show that for every integer $r \geq 1$ and any $\varepsilon > 0$ there exist $c \geq 0$ and a finite connected cyclically reduced Γ-graph Δ such that for every nondegenerate subtree $K \subseteq X$ of radius $\leq r$ we have $|\langle K, \mu\rangle_\alpha - \langle K, c\mu_\Delta\rangle| < \varepsilon$.

Choose a large integer $r \geq 1$. Define a function $\theta : \mathcal{B}_{\Gamma,r} \to \mathbb{R}_{\geq 0}$ by putting $\theta(K) = \langle K, \mu\rangle_\alpha$. Then $\theta \in \mathcal{Q}_{\Gamma,r}$. Since the polyhedron $\mathcal{Q}_{\Gamma,r}$ is defined by a finite collection of linear equations and inequalities with rational (actually, integer) coefficients, the points with rational coordinates are dense in $\mathcal{Q}_{\Gamma,r}$. Thus we can find a nonzero $\theta' \in \mathcal{Q}_{\Gamma,r}$ such that for every $K \in \mathcal{B}_{\Gamma,r}$ $\theta'(K) \in \mathbb{Q}$ and $|\theta'(K) - \theta(K)|$ is arbitrarily small.

In view of Corollary 3.5, if $\mu' \in \mathcal{S}\mathrm{Curr}(F_N)$ is such that $\theta'(K) = \langle K, \mu'\rangle_\alpha$ for every $K \in \mathcal{B}_{\Gamma,r}$ then for every finite subtree $K \subseteq X$ of radius $\leq r$ the value $|\langle \mu, K\rangle_\alpha - \langle \mu', K\rangle_\alpha|$ is also arbitrarily small.

Choose an integer $m \ge 1$ such that for every $K \in \mathcal{B}_{\Gamma,r}$ we have $m\theta'(K) \in \mathbb{Z}$ and put $\theta'' := m\theta'$. By Theorem 4.3, there exists a finite cyclically reduced Γ-graph Δ such that $\langle K, \mu_\Delta \rangle_\alpha = \theta''(K) = m\theta'(K)$ for every $K \in \mathcal{B}_{\Gamma,r}$.

Let $\Delta_1, \dots, \Delta_s$ be the connected components of Δ. Put $\mu' := \frac{1}{m}\mu_\Delta = \sum_{i=1}^s \frac{1}{m}\mu_{\Delta_i}$. Thus each $\frac{1}{m}\mu_{\Delta_i}$ is rational and hence μ' belongs to the linear span of the set of all rational subset currents in $\mathcal{S}\mathrm{Curr}(F_N)$. By Proposition 5.2 of [9], the set $\mathcal{S}\mathrm{Curr}_r(F_N)$ of all rational currents is dense in its linear span in $\mathcal{S}\mathrm{Curr}(F_N)$. (Note that the proof of Proposition 5.2 in [9] was based on an explicit combinatorial surgery argument using large finite covers and did not rely on the results of Bowen and Elek about unimodular graph measures). Therefore there exists $\mu'' \in \mathcal{S}\mathrm{Curr}_r(F_N)$ such that $|\langle K, \mu' \rangle_\alpha - \langle K, \mu'' \rangle|$ is arbitrary small for all finite subtrees $K \subseteq X$ of radius $\le r$.

It follows that for every finite subtree $K \subseteq X$ of radius $\le r$ the value $|\langle \mu, K \rangle_\alpha - \langle \mu'', K \rangle_\alpha|$ is also arbitrarily small, as required.

$\square$

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